8.1 Laplace's equation

Laplace's equation for a function of one, two or three variables is defined as

$$\nabla^2 u(x) = \frac{d^2}{dx^2} u(x) = 0$$
$$\nabla^2 u(x,y) = \frac{\partial^2}{\partial x^2} u(x,y) + \frac{\partial^2}{\partial y^2} u(x,y) = 0$$
$$\nabla^2 u(x,y,z) = \frac{\partial^2}{\partial x^2} u(x,y,z) + \frac{\partial^2}{\partial y^2} u(x,y,z) + \frac{\partial^2}{\partial z^2} u(x,y,z) = 0$$

or more succinctly as $\nabla^2 u = 0$. Force is proportional to the second derivative, and Laplace's equation is a statement that at any point, all forces cancel each other out.

For example, in one dimension, the string of a guitar that is taught between two points, that string adopts the most stable configuration: a straight line. (To be fair, there will be a slight dip due to the force of gravity pulling down on the string, but that is a negligible effect.)

For example, in two dimensions suppose we have a wire forming a closed loop, and we dip that wire into a soap solution. That soap solution will create a film that quickly stabilizes on a shape that no longer moves: that is, at each point, the forces balance each other out. If you apply a force, blowing on it, the film distorts, but with the removal of that forcing function, the solution moves back to the stable solution.

In general, we define a boundary problem where a quantity is described along a one-, two- or three-dimensional closed boundary, and want to approximate the solution in the region enclosed in the boundary.

In one dimension, the solution to Laplace's equation is trivial:

$$\nabla^2 u(x) = \frac{d^2}{dx^2} u(x) = 0$$
$$u(a) = u_a$$
$$u(b) = u_b$$

The class of all functions that have a second derivative equal to zero is all linear polynomials, and the only linear polynomial that interpolates the two points (a, u_a) and (b, u_b). That polynomial is

$$\frac{u_b(x-a) - u_a(x-b)}{b-a}.$$

In two dimensions, however, it becomes more difficult.

$$\frac{\partial^2}{\partial x^2} u(x,y) + \frac{\partial^2}{\partial y^2} u(x,y) = 0.$$

Now, we have

$$\frac{\partial^2}{\partial x^2}u(x,y) = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + \frac{1}{12}\frac{\partial^4}{\partial x^4}u(\xi,y)h^2$$
$$\frac{\partial^2}{\partial y^2}u(x,y) = \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2} + \frac{1}{12}\frac{\partial^4}{\partial y^4}u(x,\upsilon)h^2$$

We can substitute these into the equation:

$$\frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2} \approx 0\,.$$

Note that we can multiply through by h^2, and collect similar terms:

$$\frac{u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h)}{4} \approx u(x,y)\,.$$

What this says is that the value of the function at the point (x, y) is approximately the average of the value of the function surrounding it.

Background

For a function of two variables, the first and second partial derivatives may be approximated by

$$\frac{\partial}{\partial x}u(x,y) = \frac{u(x+h,y) - u(x-h,y)}{2h} + \frac{1}{6}\frac{\partial^3}{\partial x^3}u(\xi,y)h^2$$
$$\frac{\partial}{\partial y}u(x,y) = \frac{u(x,y+h) - u(x,y-h)}{2h} + \frac{1}{6}\frac{\partial^3}{\partial y^3}u(x,\upsilon)h^2$$
$$\frac{\partial^2}{\partial x^2}u(x,y) = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + \frac{1}{12}\frac{\partial^4}{\partial x^4}u(\xi,y)h^2$$
$$\frac{\partial^2}{\partial y^2}u(x,y) = \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2} + \frac{1}{12}\frac{\partial^4}{\partial y^4}u(x,\upsilon)h^2$$

The gradient is defined as a vector of the partial derivatives of a function:

$$\nabla u(x) = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x}u(x) \end{pmatrix}$$
$$\nabla u(x,y) = \begin{pmatrix} \frac{\partial}{\partial x}u(x,y) \\ \frac{\partial}{\partial y}u(x,y) \end{pmatrix}$$
$$\nabla u(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial x}u(x,y,z) \\ \frac{\partial}{\partial y}u(x,y,z) \\ \frac{\partial}{\partial z}u(x,y,z) \end{pmatrix}$$

The Laplacian is defined as the inner product of the gradient operator with itself, so $\sqrt{a^2 - b^2}$

$$\nabla^2 u(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x)$$
$$\nabla^2 u(x, y) = \frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y)$$
$$\nabla^2 u(x, y, z) = \frac{\partial^2}{\partial x^2} u(x, y, z) + \frac{\partial^2}{\partial y^2} u(x, y, z) + \frac{\partial^2}{\partial z^2} u(x, y, z)$$

Acknowledgments